Series I, exercise 1 Prove that there does not exist an arithmetical sequence such that its three consecutive terms are also different terms of the sequence (2^n).

Solution Suppose $2^k, 2^l, 2^m$ form an arithmetic progression, where $0 < k < l < m$. Then $2^l - 2^k = 2^m - 2^l$, and therefore $2^{l+1} = 2^k(2^{m-k} + 1)$. Because of $k < l < m$, we have $m \geq k + 2$, and so $2^{m-k} + 1$ is an odd number greater than or equal to 5. This odd number must divide 2^{l+1}. This is a contradiction.

Series I, exercise 2 What is the greatest and the smallest number (not starting from 0), consisting of all digits from 0 to 9 (where every digit appears exactly once) and is divisible by 11?

Solution The number is divisible by 11 if and only if the difference between the sum of digits on odd places and the sum of digits on even places is divisible by 11. First, we look for the greatest number x consisting of all digits which is divisible by 11. Denote by a the sum of digits on odd places of x and by b the sum of digits on even places of x. Observe that the sum of all digits we want to use is 45. Hence the number $a - b = (a + b) - 2b = 45 - 2b$ must be odd. Since $a, b \geq 0 + 1 + 2 + 3 + 4 = 10$ and $a, b \leq 9 + 8 + 7 + 6 + 5 = 35$, we have $|a - b) \leq 25$. Therefore, $a - b = 11$ or $a - b = -11$. Since also $a + b = 45$, we have $a = 28$, $b = 17$ or $a = 17$, $b = 28$. Let A be the set of digits on odd places and B be the set of digits on even places of x. In x there are digits from A and B in turns and the digits from both sets should be taken in decreasing order. The optimal set A would consist of $9, 7, 5, 3, 1$. Then $x = 9876543210$. But $9 + 7 + 5 + 3 + 1 = 25$ and should be equal to 28. We would like to modify A by changing the smaller numbers (then we change the last digits of the number). Hence we replace 1 in A by 4. Finally, $x = 9876524130$.

First, we look for the smallest number y consisting of all digits which is divisible by 11. Let a, b, A, B be as above. Since we need to start from 1, 0 should be in B. Moreover a should be equal to 17 (as we want the smallest number). We have $1 + 2 + 3 + 4 + 5 = 15$, so we need to make it greater by 2. We would like to change the smaller numbers to have them in B. But we can only change 4 to 6. So, $y = 1024375869$.

Series I, exercise 3 Let $M \subset \mathbb{N}$ be a set such that for any $n, m \in M$

$$n > m \Rightarrow n - m \geq \frac{nm}{25}. \tag{1}$$

What is the maximal number of elements of the set M?

Solution

First, observe that if $n, m \geq 25$, then we have $n - m < n \leq \frac{nm}{25}$, so the desired inequality is not satisfied. Hence there can be only one $n \in M$ which is greater than 24. We will construct the maximal set $M = \{n_1, n_2, \ldots, n_k\}$ for some $k \in \mathbb{N}$, where $n_i < n_{i+1}$ for all $i < k$, satisfying condition (1). To maximize the number of elements of the set M, we should take n_i as small as possible. In particular, take $n_1 = 1$.

For $i < k$, let $d_i := n_{i+1} - n_i$. By (1),

$$d_i \geq \frac{n_i n_{i+1}}{25} = \frac{n_i(n_i + d_i)}{25},$$

which is equivalent to

$$(25 - n_i)d_i \geq n_i^2,$$

and thus to

$$d_i \geq \frac{n_i^2}{25 - n_i},$$

because $n_i < 25$. We have $d_1 \geq \frac{1}{24} \in (0,1)$, so we can take $n_2 = n_1 + 1 = 2$. Similarly,

$$d_2 \geq \frac{4}{23} \in (0,1), \text{ so } n_3 = n_2 + 1 = 3,$$

$$d_3 \geq \frac{9}{22} \in (0,1), \text{ so } n_4 = n_3 + 1 = 4,$$

$$d_4 \geq \frac{16}{21} \in (0,1), \text{ so } n_5 = n_4 + 1 = 5,$$

$$d_5 \geq \frac{25}{20} \in (1,2), \text{ so } n_6 = n_5 + 2 = 7,$$

$$d_6 \geq \frac{49}{18} \in (2,3), \text{ so } n_7 = n_6 + 3 = 10,$$

$$d_7 \geq \frac{100}{15} \in (6,7), \text{ so } n_8 = n_7 + 7 = 17,$$

$$d_8 \geq \frac{289}{8} \in (36,37) \text{ so } n_9 = n_8 + 37 = 54.$$

As there cannot be two numbers in M greater than 25, we cannot add any more numbers. Now, it suffices to check that such defined M satisfies (1). Let $g : (0,\infty) \times (0,\infty) \to \mathbb{R}$ be given by the formula $g(n,m) = n - m - \frac{nm}{25}$. We want to show that for all $n, m \in M$, $n > m$, we have $g(n, m) > 0$. Let $n \in M$. Since g is decreasing with respect to m, to show that $g(n, m) > 0$ for all $m < n$, it suffices to show that $g(n, m) > 0$ for the greatest $m \in M$ such that $m < n$. However, this means that we only need to check if $g(n_i, n_{i+1}) > 0$ for $i < 9$, but this has been already proved. Hence M satisfies (1) and has maximal number of elements, which is 9.

Series I, exercise 4 For $n \in \mathbb{N}$ solve the following equation in integers:

$$\left(2-\frac{1}{x_1}\right)\left(2-\frac{1}{x_2}\right)\dots\left(2-\frac{1}{x_n}\right)=3.$$

Solution Case $n=1$

$$2-\frac{1}{x_1}=3$$
$$x_1=-1.$$

Case $n=2$

$$\left(2-\frac{1}{x_1}\right)\left(2-\frac{1}{x_2}\right)=3.$$

Equivalently we have

$$x_1x_2-2x_2-2x_1+4=3,$$

and so

$$x_2(x_1-2)-2(x_1-2)=3,$$
$$(x_1-2)(x_2-2)=3.$$

The only solutions in integers are: $(-1,1)$, $(1,-1)$, $(3,5)$, $(5,3)$.

Case $n\geq 3$. If $x_k\neq 1$, then $2-\frac{1}{x_k}\geq\frac{3}{2}$. By s denote number of elements x_k which are not equal to 1. We have

$$\left(2-\frac{1}{x_1}\right)\left(2-\frac{1}{x_2}\right)\cdot\ldots\cdot\left(2-\frac{1}{x_n}\right)\geq\left(\frac{3}{2}\right)^s\cdot 1^{n-s}$$

Therefore,

$$3\geq\left(\frac{3}{2}\right)^s$$

It implies $s\leq 2$. By case $n=2$, we obtain that solutions are of the form

$$(x_1,x_2,\dots,x_n)=(1,1,\dots,a,b)$$

where $(a,b)\in\{(-1,1),(1,-1),(3,5),(5,3)\}$ and all permutations.)

Series I, exercise 5 Fix $n \in \mathbb{N} \setminus \{1\}$. Two players play the following game. Starting from $k = 2$, each player has two possible moves: he can replace k by $k + 1$ or by $2k$. The player who is forced to choose a number greater than n loses. For each n which player (the first one or the second one) has a winning strategy, that is, a strategy which use will ensure the win no matter what the other player will do?

Solution

First, observe that if n is odd, then the first player has a winning strategy: he just has to add 1 in all of his turns. After each of his turns k will be odd. Indeed, after the first turn we have $k = 3$, so an odd number. Moreover, if k is odd, then $k+1$ and $2k$ are even, so after the second player's turn k is even. And then $k + 1$ will be odd, so it cannot be greater than n if k is not greater than n. So, the first player will win.

If $n = 2$, then the second player obviously wins. If $n = 4$ or $n = 6$, then the first player should choose $k = 4$ in his first turn. Then the second player is forced to choose $k = 5$ (and loses if $n = 4$). If $n = 6$, then the first player take $k = 6$ and wins.

We will now show that if some player X has a winning strategy for some n, then he also has a winning strategy for $4n$ and $4n + 2$. Indeed, using strategy for n, the player X is sure that the other player will be first to choose a number k greater than n. Then, the player X chooses $2k$ and this number is greater than $2n + 1$. So, now the second player will lose if he multiplies k by 2, so he has to add one. Player X do the same, and so he always choose even numbers, so he will win, because he eventually will reach $4n$ or $4n + 2$.

Finally, we have that if the first player has a winning strategy for n, then he also has for $4n, 4n + 1, 4n + 2, 4n + 3$. So, the only case when the second player wins is if $n = 2$ or if $n = 4p$ or $n = 4p + 2$ for some p for which we know that the second player has a winning strategy (that is, the second player wins for 2, $4 \cdot 2 = 8$, $4 \cdot 2 + 2 = 10$, $4 \cdot 8 = 32$, $4 \cdot 8 + 2 = 34$, and so on).

Series II, exercise 1 Let A be a smallest set of natural number such that

(1) $2 \in A$;

(2) $n^2 \in A \Rightarrow n \in A$

(3) $n \in A \Rightarrow n + 5 \in A$.

Which numbers belong to A, and which do not?

Solution From (1) and (3) we immediately get that $A \subset B$, where $B := \{5k+2 \colon k \in \mathbb{N}\}$. We will show that $A = B$. Of course, B satisfies (1) and (3). We will show that it also satisfies (2). All elements of B are equal to 2 mod 5. We will show that there is $n \in \mathbb{N}$ such that $n^2 =_{mod\,5} 2$. We will consider all 5 possible cases:

$$n =_{mod\,5} 0 \Rightarrow n^2 =_{mod\,5} 0,$$

$$n =_{mod\,5} 1 \Rightarrow n^2 =_{mod\,5} 1,$$

$$n =_{mod\,5} 2 \Rightarrow n^2 =_{mod\,5} 4,$$

$$n =_{mod\,5} 3 \Rightarrow n^2 =_{mod\,5} 4,$$

$$n =_{mod\,5} 4 \Rightarrow n^2 =_{mod\,5} 1,$$

so $n^2 \neq_{mod\,5} 2$ for any $n \in \mathbb{N}$ and so (2) is satisfied for B (as the first part of the implication is always false).

Series II, exercise 2 On the table there are n glasses which are put upside down. Is it possible to obtain, after some number of moves, a setting in which all glasses are put in upright position if in every move we turn exactly m glasses ($m < n$)? If the answer is positive, what is the minimal number of moves to do that? Give the answer depending on n and m.

Solution First, observe that if $m = n$, then one move suffices to turn all glasses. Hence we assume that $m < n$. For $k \leq n$ we will denote by (k) a state in which we have exactly k glasses in upright position. So, we start from the state (0), and we want get to the state (n).

Suppose that we have a state (k). We will examine what are possible states after the next move. If $k - m \geq 0$ and $k + m \leq n$, then after the next move we can maximally turn m upside down glasses, receiving the state $(k + m)$. If we turn $m - 1$ upside down glasses and one upright glass, we will obtain the state $(k + m - 1 - 1) = (k + m - 2)$. Further, if we turn exactly j upside down glasses and $m - j$ upright glasses (for $j \in \{0, \dots, m\}$), then we receive a state $(k + j - m + j) = (k + m - 2j)$. So, the only possible states are: $(k-m), (k-m+2), \dots, (k+m-2), (k+m)$. If $k+m > n$, then we can maximally turn $n-k$ upside down glasses, so the highest possible state is $(k+n-k-(m-(n-k))) = (2n-m-k)$. Similarly, if $k - m < 0$, then we can maximally turn k upright glasses, so the lowest possible state which we can obtain is $(k + (m - k) - k) = (m - k)$.

Observe that if m and k has the same even parity, then in the next move we can obtain only even number of upright glasses, and otherwise we can only have odd number of upright glasses. After the first move we will always be in the state (m), and the penultimate state must be $(n - m)$. So, there are two important factors which has an impact on the solution: the even parity of n and m and whether $m + m \leq n$. So, let us consider the following cases.

1. m, n- even, $2m > n$. Then $n - m$ is even and $n - m \leq 2n - m - m = 2n - 2m$, therefore from the state (m) we can get to the state $(n - m)$ in one move. Hence the optimal way from (0) to (n) is following:

$$(0) \to (m) \to (n - m) \to (n).$$

Thus, we need exactly 3 moves.

2. m, n- even, $2m \leq n$. Let $k, r \in \mathbb{N}, r < m$ be such that $n = mk + r$. Suppose that $r > 0$. Then r is even. So it is possible to get from (m) to $(m + r)$ in one move. Hence the optimal way from (0) to (n) is following:

$$(0) \to (m) \to (m + r) \to (2m + r) \to \cdots \to (km + r) = (n).$$

We need $k+1 = \lceil \frac{n}{m} \rceil$ moves, where $\lceil x \rceil = \min\{y \in \mathbb{Z} \colon y \geq x\}$. If $r = 0$, then it suffices to make $k = \frac{n}{m}$ moves. Generally, we need $\lceil \frac{n}{m} \rceil$ moves.

3. n-even, m-odd, $2m > n$. Then $n-m$ is odd, so we can get the state $(n-m)$ only after odd number of moves. Observe that from the state (m) we can get to states: $(0), (2), \ldots (2n-m-m) = (2n-2m)$. From each we can return to (m), however we would like to get to $(n-m)$, but $n-m < m$. The lowest state we can get from the state (k) is $(m-k)$, so the larger k, the more possible states we can get in the following move. Hence in the second step we should get from (m) to the state $(2n-2m)$. If $n-m \geq m-(2n-2m) = 3m-2n$, then we can get from $(2n-2m)$ to $(n-m)$. If not, then in the next even move (fourth) we should choose the greatest possible state. And the possible maximum (that is, $2n-m-k$) is the largest for the smallest k. So from the state $(2n-2m)$ we should go to the lowest possible state, that is, $(3m-2n)$. Then we again choose the maximal possible state, that is, $(2n-m-3m+2n) = (4n-4m)$. If $n-m \geq m-(4n-4m) = 5m-4n$, then we can get from $(4n-4m)$ to $(n-m)$. If not then we continue the process described above.

We will inductively prove that, until the end of the process, after step $2k$ we will have a state $(2kn-2km)$, and in the move $2k+1$ we will get to state $((2k+1)m-2kn)$. Assume that after step $2k$ we have a state $(2kn-2km)$. In the next state we choose the lowest possible state, that is, $(m-(2kn-2km)) = ((2k+1)m-2kn)$. In the step $2k+2$ we choose the largest possible state, that is, $(2n-((2k+1)m-2kn)-m) = ((2k+2)n-(2k+2)m)$. By induction, we have the assertion.

We will now show that this process will finish after finite number of steps. It suffices to notice that $(2k+3)m-(2k+2)n < (2k+1)m-2kn$, so the states obtained in the odd steps are smaller and smaller. Hence we continue the process until k such that $(2k+1)m-2kn \leq n-m$.

Let $k, r \in \mathbb{N}, r < n-m$ be such that

$$2k(n-m)+r = n.$$

Suppose that $r > 0$. Then $r = n-2k(n-m) = 2km-(2k-1)n < n-m$. Therefore, $(2k+1)m-2kn < 2km-(2k-1)n < n-m$, (because $n > m$), but in the same time $(2k-1)m-(2k-2)n = 2km-(2k-1)n+n-m \geq n-m$. So, the optimal path from (0) to (n) is following:

$$(0) \to (m) \to (2n-2m) \to (3m-2n) \to (4n-4m)$$

$$\to (5m-4n) \to \cdots \to (2kn-2km) \to (n-m) \to (n).$$

The necessary number of steps is $2k+2 = 2 \cdot \lceil \frac{n}{2(n-m)} \rceil$. If $r = 0$, then the optimal path is:

$$(0) \to (m) \to (2n-2m) \to (3m-2n) \to (4n-4m) \to (5m-4n) \to \dots \to (2kn-2km) = (n).$$

We need $2k = 2 \cdot \frac{n}{2(n-m)}$ moves. Finally, it is necessary to make $2 \cdot \lceil \frac{n}{2(n-m)} \rceil$ moves.

4. n-even, m-odd, $2m \le n$. Then we act as in case 2., considering $2m$ instead of m. Let $k, r \in \mathbb{N}$, $r < 2m$ be such that $n = 2mk + r$. Suppose that $r > 0$. Then r is even, so we can get from (r) to (m). Hence the optimal path is:

$$(0) \to (m) \to (r) \to (m+r) \to (2m+r) \to \dots (2mk+r) = (n).$$

We need $2k+2 = 2 \cdot \lceil \frac{n}{2m} \rceil$ moves. If $r = 0$, the optimal path is:

$$(0) \to (m) \to (2m) \to \dots (2mk) = (n).$$

We need $2k = 2 \cdot \frac{n}{2m}$ steps. Finally, we have to make $2 \cdot \lceil \frac{n}{2m} \rceil$ moves.

5. n- odd, m - even. Since m is even, then we can only obtain even states, so there is no way to get from (0) to (n).

6. n, m -odd, $2m > n$. Then $n - m$ is even and $n - m < 2n - m - m = 2n - 2m$, so we can obtain a state $(n-m)$ after one move from (m). So, the optimal way is:

$$(0) \to (m) \to (n-m) \to (n).$$

The required number of steps is 3.

7. n, m -odd, $2m \le n$. Let $k, r \in \mathbb{N}, r < m$ be such that $n = mk + r$. Then either r is even or $r + m$ is even, so one of these states can be obtained directly from (m). If $r + m$ is even, then the optimal path from (0) to (m) is following:

$$(0) \to (m) \to (m+r) \to (2m+r) \to \dots \to (km+r) = (n).$$

We need $k+1$ moves. If r is even and $r > 0$, then the possible path is:

$$(0) \to (m) \to (r) \to (m+r) \to \dots \to (km+r) = (n).$$

It needs $k+2$ moves. However, we would need to get back from (m) do (r). But $(m+r)$ is odd, so we could not obtain this state directly from (m). If we would get to $(2m)$, then in the next step we could only get to odd states. So, it would be impossible to get to $(2m+r)$, however we could get back to $(m+r)$. But this would not let us eventually get to (n) after less number of steps. So, the suggested path is optimal. If $r=0$, then the optimal path is:

$$(0) \to (m) \to (2m) \to \cdots \to (km) = (n).$$

We need then k steps. Finally, we have to make $\frac{n}{m}$ steps if $\frac{n}{m} \in \mathbb{N}$. If $\frac{n}{m} \notin \mathbb{N}$, then we need $\lceil \frac{n}{m} \rceil$ moves if $\lceil \frac{n}{m} \rceil$ is odd, and $\lceil \frac{n}{m} \rceil + 1$ moves if $\lceil \frac{n}{m} \rceil$ is even.

Series II, exercise 3 There is some number of equilateral triangles on the plane. Together they cover a surface of area equal to 1. Show that it is possible to choose from these triangles some number of disjoint triangles such that the sum of their areas is not less than $\frac{1}{16}$.

Solution Let us calculate the area of ϵ-neighbourhood of an equilateral triangle with side a. This neighbourhood consists of the triangle itself of area $\frac{\sqrt{3}}{4}a^2$, three rectangles of area ϵa each and three parts of a disk that together give a whole area or a disk with radius ϵ. Thus, this area sums to $\frac{\sqrt{3}}{4}a^2 + 3\epsilon a + \pi\epsilon^2$.

Let us choose the biggest triangle and denote it's side by a_1. If its area is $\geq \frac{1}{16}$ we are done. Assume to the contrary that $a_1^2 < \frac{1}{4\sqrt{3}}$. All the triangles that have a nonempty intersection with this one lie in it's a_1-neighbourhood whose area is $(\frac{\sqrt{3}}{4} + 3 + \pi)a_1^2$ and thus is smaller than $(\frac{\sqrt{3}}{4} + 3 + \pi)\frac{1}{4\sqrt{3}} < 1$. Because of that there have to exist triangles that are disjoint with the chosen triangle. Let a_2 be a side of such a triangle. If sum of areas of first two triangles is greater than $\frac{1}{16}$ we are done. If not, we repeat the previous argument for a_2. Thus, at every step we find triangles which are pairwise disjoint and whose sum of areas is ≥ 16 or we find a triangle that is disjoint from all the others. Because the number of triangles is finite the proof is complete.

Series II, exercise 4 Is there an injective function $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ the following inequality is satisfied

$$f(x^2) - (f(x))^2 \geq \frac{1}{4}?$$

Solution Firstly, observe that for $x = 0$ and $x = 1$ we obtain

$$f(0) - f^2(0) \geq \frac{1}{4}$$
$$f(1) - f^2(1) \geq \frac{1}{4}$$

Inequality $t^2 - t + \frac{1}{4} \leq 0$ has only one solution $t = \frac{1}{2}$. It means $f(0) = f(1) = \frac{1}{2}$. It contradicts the injectivity of the function f. Therefore, there does not exist such a function.

Series II, exercise 5 Let k and n be natural numbers satisfying

$$k^{n^k} = n^{k^n}.$$

Prove that $k = n$.

Solution Let $k, n \in \mathbb{N}$ be such that $k^{n^k} = n^{k^n}$. We have $k = 1$ if and only if $n = 1$.

If $k = 2$, then $n > 1$. If $n > 2$, then

$$2^{n^2} = n^{2^n} > 2^{2^n}.$$

So, $n^2 > 2^n$. However, for $n \geq 4$ we obviously have $2^n \geq n^2$. So, since $n^2 > 2^n$, we must have $n < 4$. For $n = 3$ we would have an equality $2^{3^2} = 3^{2^3}$ which is false. Therefore, $n = 2$. Similarly, if $n = 2$, then $k = 2$.

Suppose that $n, k \geq 3$ and $n \neq k$. Without loss of generality we can assume that $n < k$. Consider a function $f\colon (0, +\infty) \to \mathbb{R}$ given by the formula $f(x) = \frac{\ln x}{x}$. Then $f'(x) = \frac{1-\ln x}{x^2}$. We have $f'(x) < 0$ for $x > e$, so the function f is decreasing on the interval $(e, +\infty)$. Hence

$$\frac{\ln n}{n} > \frac{\ln k}{k},$$

and so

$$k \ln n > n \ln k$$

$$\ln n^k > \ln k^n$$

$$n^k > k^n.$$

In the consequence,

$$k^{n^k} > n^{n^k} > n^{k^n},$$

a contradiction. Thus, $n = k$.

Series III, exercise 1 We have a deck consisting of $2n$ cards. After a shuffling the order of cards changes from $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ to $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$. For what n cards will return to the initial order after shuffling them 8 times?

Solution

First, observe that the first and the last card never changes their place. Let us enumerate the positions of the card starting from 0 and finishing at $2n-1$ (the card a_1 is on the position 0, a_2 on the position 1 and so on, and b_n is on the position $2n-1$). Shuffling the cards once moves the card from position i to the position $2i$ if $i < n$, and to the position $2i - (2n-1)$ if $i \geq n$. We can ignore the card b_n as it does not change its position. Then we can see that generally the card from the position i moves to the position $2i \, mod \, (2n-1)$. So, after shuffling the deck 8 times, the card from the position i moves to the position $256i \, mod \, (2n-1)$. So, the deck gets back to the initial state if and only if $256i =_{mod \, 2n-1} i$ for all $i < 2n-1$. This is equivalent to the equality $256 =_{mod \, 2n-1} 1$, which holds if and only if $(2n-1)|255$. The divisors of 255 are: $1, 3, 5, 15, 17, 51, 85, 255$, so the possible n are: $1, 2, 3, 8, 17, 26, 43, 128$.

Series III, exercise 2 Does there exist an infinite increasing sequence of prime numbers (q_n) such that for all $n \in \mathbb{N}$

$$q_{n+1} \geq \frac{q_n + q_{n+2}}{2} \text{ ?}$$

Solution Let us imagine that (q_n) is a sequence where for each n q_n is prime. Then

$$\begin{aligned} q_{n+1} &\geq \frac{q_n + q_{n+2}}{2} \\ 2q_{n+1} &\geq q_n + q_{n+2} \\ q_{n+1} - q_n &\geq q_{n+2} - q_{n+1}. \end{aligned} \tag{2}$$

Let's define the sequence of differences d_i as $d_{i+1} = q_{i+1} - q_i$. Note that $\forall_i \; d_i \in \mathbb{Z}$ and $1 < d_i \leq d_{i-1}$. Obviously (d_n) must become constant as a nonincreasing sequence of positive integers. Let us denote by z the limit of this sequence. Let us analize the remaiders modulo $z-1$ of the numbers in the sequence (q_n). For n big enough (such that $q_{n+1} - q_n = z$) we have $q_n \equiv a \mod (z-1)$, $q_{n+1} = q_n + z \equiv a + 1 \mod (z-1)$ and so on. As the remainders increase at some point we obtain $q_m \equiv 0 \mod (z-1)$ and so q_m is not prime, a contradiciton. This ends the proof.

Series III, exercise 3 For $x > 0$ let

$$g(x) = \lim_{r \to 0} \left((x+1)^{r+1} - x^{r+1}\right)^{\frac{1}{r}}.$$

Calculate $\lim_{x \to \infty} \frac{g(x)}{x}$.

Solution

First, observe that for $r > -1$ and any positive x, we have $(x+1)^{r+1} - x^{r+1} > 0$. Thus, by the continuity of the logarithm,

$$\ln g(x) = \lim_{r \to 0} \ln((x+1)^{r+1} - x^{r+1})^{1/r} = \lim_{r \to 0} \frac{1}{r} \ln((x+1)^{r+1} - x^{r+1})$$

By L'Hôpital's rule, we obtain

$$\ln g(x) = \lim_{r \to 0} \frac{(x+1)^{r+1}\ln(x+1) - x^{r+1}\ln x}{(x+1)^{r+1} - x^{r+1}}$$

$$= \frac{(x+1)\ln(x+1) - x\ln x}{(x+1) - x} = \ln((x+1)^{x+1}x^{-x})$$

Hence

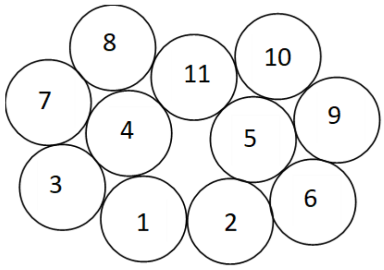
$$g(x) = (x+1)^{x+1}x^{-x} = (x+1)\left(1 + \frac{1}{x}\right)^x$$

Therefore,

$$\lim_{x \to \infty} \frac{g(x)}{x} = \lim_{x \to \infty} \frac{x+1}{x}\left(1 + \frac{1}{x}\right)^x = 1 \cdot e = e.$$

Series III, exercise 4 There are n identical balls on the table. We colour them in such a way that balls which are contacting have different colors. What is the smallest number of balls n such that there exists a setting of n balls in which it is necessary to use at least 4 colors to colour balls?

Solution



First, we will prove that in the setting on the picture above, which uses 11 balls, we need at least 4 colors to colour balls. Suppose that we can color them using 3 colors A, B, C. We use color A for colouring ball 11. Then we use colors B, C to colour balls 8 and 4, respectively. Then ball 7 must have color A. Then ball 3 has color B and 1 has color A. Analogously, we show that ball 2 has to have color A (it does not matter whether ball 10 will have color B or C). Therefore, 1 and 2 has the same color, a contradiction.

If there was another setting with less than 11 balls, in which we need to use four colors, then in this setting there are 2 contacting balls which would have to have the same color if we would like to use 3 colors in that setting (like balls 1 and 2 on the picture). Since these balls must have the same color, there must be at least 2 balls contacting each of them. Thus, we have balls 1-6 like on the picture, but angles of their setting may be different. In particular, balls 4 and 5 could be contacting. However, in every setting of 6 balls like that we can use 3 colors. Thus, we need to add another balls (7, 9) in such a way that pairs of balls 3, 4 and 5, 6 has to be coloured using the same two colors. Still, we can use 3 colors, so we need to make sure that balls 7 and 9 has the same color, so we add 2 balls: one contacting 7 and 4, and one contacting 5 and 9 (that is, balls 8 and 10). However, even if balls 8 and 10 are contacting, we can easily use only 3 colors. Hence, we cannot use less than 11 balls.

Series III, exercise 5 Let $A = (a_{ij})$ be a real matrix fo dimensions $m \times n$ with at least one non-zero element. For $i \in 1, 2, \ldots, m$ let $W_i = \sum_{j=1}^n a_{ij}$ be a sum of terms from i-th row of the matrix, and for $j \in 1, 2, \ldots, n$ let $K_i = \sum_{i=1}^m a_{ij}$ be a sum of terms from j-th column of the matrix A. Prove that there exist $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$ such that

$$(a_{kl} > 0) \wedge (W_k \geq 0) \wedge (K_l \geq 0)$$

or

$$(a_{kl} < 0) \wedge (W_k \leq 0) \wedge (K_l \leq 0).$$

Solution

Denote

$$I^+ := \{i \leq m \colon W_i \geq 0\},$$

$$I^- := \{i \leq m \colon W_i < 0\},$$

$$J^+ := \{j \leq n \colon K_i > 0\},$$

$$J^- := \{j \leq n \colon K_i \leq 0\}.$$

Suppose that the assertion is false. Then, for $(i, j) \in I^+ \times J^+$ we must have $a_{ij} \leq 0$ and for $(i, j) \in I^- \times J^-$ we must have $a_{ij} \geq 0$. We have

$$\sum_{(i,j) \in I^- \times J^+} a_{ij} = \sum_{i \in I^-} (\sum_{j=1}^n a_{ij} - \sum_{j \in J^-} a_{ij}) = \sum_{i \in I^-1} W_i - \sum_{(i,j) \in I^- \times J^-} a_{ij} \leq 0.$$

On the other hand,

$$\sum_{(i,j) \in I^- \times J^+} a_{ij} = \sum_{j \in J^+} (\sum_{i=1}^m a_{ij} - \sum_{i \in I^+} a_{ij}) = \sum_{i \in J^+} K_j - \sum_{(i,j) \in I^+ \times J^+} a_{ij} \geq 0.$$

So, $\sum_{(i,j) \in I^- \times J^+} a_{ij} = 0$, and hence, by the calculations above, $\sum_{i \in I^-1} W_i = \sum_{i \in J^+} K_J = 0$. Therefore, $I^- = J^+ = \emptyset$. So, $W_i \geq 0$ for all $i \leq m$ and $K_j \leq 0$ for all $j \leq n$. Thus,

$$0 \leq \sum_{i=1}^m W_i = \sum_{j=1}^n K_j \leq 0,$$

and so $W_i = 0$ for all $i \leq m$ and $K_j = 0$ for all $j \leq n$. Since there are $k \leq m, l \leq n$ such that $a_{kl} \neq 0$ and $W_k = K_l = 0$, we obtain a contradiction with the assumption that the assertion does not hold, which finishes the proof.